Some aspects of the synchronization in coupled maps

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We numerically study the synchronization behavior of a coupled map lattice consisting of a chain of chaotic logistic maps exhibiting power law interactions. We report two main results. First, we find a practical lower bound in the lattice size in order that this system could be considered in the thermodynamic limit in numerical simulations. Second, we observe the existence of a strong correlation between the Lyapunov dimension and the averaged synchronization time.

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Coupled map lattices (CMLs) were introduced in the literature as suitable models to study spatiotemporal behavior of spatially extended dynamical systems. Fundamentally, they are systems defined on a discrete space time and possessing continuous state variables. In the last two decades, such models have received a great, and increasing, deal of interest, which is being applied to several nonlinear phenomena including systems as diverse as physical, chemical, and biological [1]. CMLs share with real complex systems one of their most intriguing behaviors, which is the possibility of synchronization. Such a phenomenon can be observed in a great variety of real systems, going from electronic circuits to physiological processes, for example [2].

We shall be concerned here with a CML consisting of a chain of N interacting identical (chaotic) maps, located at definite sites $i=1,\ldots N$. In the absence of interactions, the maps dynamics are governed by $x_{n+1}^{(i)}=f(x_n^{(i)})$, where $x_n^{(i)}$ denote the state, or amplitude, of the map located at the site i at the discrete time $n=0,1,2,\ldots$, and we assume, without loss of generality, that $f(x)$ is some nonlinear function mapping the interval [0, 1] onto itself. At the time n the state of the whole lattice, or the *system state*, is defined as the N-dimensional vector $\mathbf{x}_n=(x_n^{(1)},x_n^{(2)},\ldots,x_n^{(N)})$. Moreover, we will be restricted to the class of CMLs whose dynamics are *linear in the map functions*, i.e., of the form

$$\mathbf{x}_{n+1}=(\mathbf{1}+\mathbf{I})\mathbf{F}(\mathbf{x}_n)\equiv\mathbf{B}\ \mathbf{F}(\mathbf{x}_n), \tag{1}$$

$$\text{with}\quad B_{ij}\geqslant 0\ \forall\ i,j\quad\text{and}\quad 0\leqslant\sum_j B_{ij}\leqslant 1\ \forall\ i, \tag{2}$$

where $\mathbf{F}(\mathbf{x})=[f(x^{(1)}),f(x^{(2)}),\ldots,f(x^{(N)})]$ and $\mathbf{I}$ is an $N\times N$ matrix giving the coupling between the maps. Conditions (2) are necessary and sufficient to hold $x_n^{(i)}\in[0,1]$ for all i, all n, and all the possible initial conditions.

To study the synchronization of the lattice, we shall use the following definition [2]. A system is said to be in a *completely synchronized state* at time n if *all the elemental maps have the same amplitude*, i.e., $x_n^{(1)}=x_n^{(2)}=\ldots=x_n^{(N)}=x_n^*$. The subspace $\mathbf{S}$ of all these states (the diagonal of the whole state space of the system) will be called *invariant*, or the *synchronization subspace*, if $\mathbf{x}_m\in\mathbf{S}$ implies that $\mathbf{x}_n\in\mathbf{S}$ for all $n>m$ [2]. The necessary and sufficient condition for $\mathbf{S}$ to be invariant is that the sum in the second of conditions (2) be the same for all rows. In this case, given the initial conditions, a *synchronized regime* starts at the instant m when (and *if*) the system state is put in $\mathbf{S}$ for the first time. This instant will be called t_s, the *synchronization time*.

Now we shall specialize the use to the case where the elemental maps are the fully chaotic logistic ones, i.e., $f(x)=4x(1-x)$, $x\in[0,1]$ [3], and the matrix $\mathbf{I}$ is symmetric, corresponding to a power law long range coupling [4], whose elements are given by $I_{ij}=\varepsilon\eta^{-1}\{r_{ij}^{-\alpha}(1-\delta_{ij})-\eta\delta_{ij}\}$, where $r_{ij}=|i-j|$ is the "distance" between sites i and j. The parameters ε and α give, respectively, the strength and the range of the interaction and $\eta=2\Sigma_r^{N'}r^{-\alpha}$ is a normalization factor, with $N'=(N-1)/2$. The parameter α can assume any nonnegative value. The extreme cases $\alpha=0$ and $\alpha\to\infty$ correspond, respectively, to *global* (mean field) and *local* (first neighbors) couplings. The first of the conditions (2) restrict the parameter ε to the unit interval [0, 1]. The second condition is automatically fulfilled, because $\Sigma_j I_{ij}=0$ for all i. With this last result the subspace $\mathbf{S}$ turns out to be invariant. Some recent works assume ε taking values outside the unit interval (see, for instance, [5]). In these cases, we must depart from the "linear" coupling (1).

Our aim in this report is to go a step further in understanding the synchronization of this system through the analysis of the synchronization time and the Lyapunov dimension. The initial state of each elemental map is assumed to be randomly chosen. Besides, we always assume odd N and periodic boundary conditions. Given N, and if the lattice is not initially synchronized, it only will synchronize after some time m if the parameters α and ε assume values inside a restricted domain in the *parameter space* [6]. In such a case the basin of attraction to the synchronization subspace turns out to be the entire state space. This domain has boundaries which can be analytically calculated from the condition $\lambda_2=0$, where λ_2 is the largest Lyapunov exponent transversal to the synchronization subspace [7,8]. These boundaries can be decomposed into an upper line, given by $\varepsilon_c'(\alpha,N)=\min\{\varepsilon_{up}(\alpha,N),1.0\}$, with $\varepsilon_{up}(\alpha,N)=3/2(1-b^{(N')}/\eta)^{-1}$, and a lower one, $\varepsilon_c(\alpha,N)=\min\{\varepsilon_{lo}(\alpha,N),1.0\}$, with $\varepsilon_{lo}(\alpha,N)$

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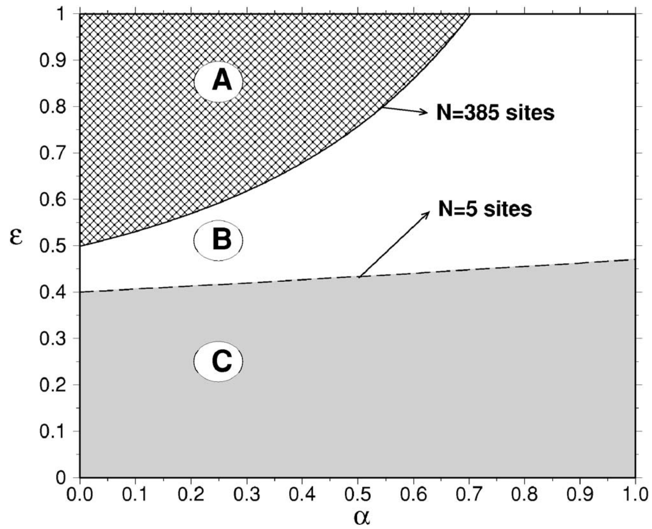

FIG. 1. Domains of synchronization in the parameter space. For $N=5$ the CML synchronizes only in regions (a) and (b); for $N=385$ it synchronizes only in region (a).

$=1/2(1-b^{(1)}/\eta)^{-1}$. Here $b^{(k)}=2\Sigma_{m=1}^{N'}\cos(2\pi km/N)m^{-\alpha}(1\leqslant k\leqslant N)$ are the eigenvalues of the circulant matrix $\widetilde{\mathbf{B}}=\eta\varepsilon^{-1}\mathbf{I}+\eta\mathbf{1}$ [7]. For instance, the domains for $N=5$ and $N=385$ are shown in Figure 1. We can observe two distinct regions for each N. For $N=385$, and if it does not start synchronized, the lattice synchronizes only in region (a), while for $N=5$ it synchronizes only in regions (a) and (b).

On the state space of the system it can be defined as the quantity $d_n=\sigma_n\sqrt{N}$, where σ_n is the standard deviation of the map amplitudes around their mean value. It turns out that this quantity corresponds, at each time n, to the *distance*, in the *state space*, from the system point $\mathbf{x}_n$ to the straight line corresponding to the synchronization subspace $\mathbf{S}$ [9]. We then use the condition $d_m=0$ as a diagnostic for the synchronization regime. Alternatively, we could use as a diagnostic the condition $\sigma_n=0$ [10]. However, due to the limited precision of numerical simulations, a quantity q will be considered equal to zero if $q<10^{-16}$ (we use double precision). Taking into account this limitation, the diagnostic based on the distance d_n turns out to be more accurate than that based on the standard deviation σ_n. For instance, if all the maps, except a single one, have the same amplitude (inside the numerical precision), the "distance criterion" says that the system is *not* synchronized, *independently* of the lattice size N. Instead, there will be a sufficiently large N for which the "standard deviation criterion" will say that the same system *is* synchronized.

Figure 2 is a plot of the averaged synchronization time $\langle t_s\rangle$ *versus* the strength parameter ε, for two values of N and α [11]. For $N=5$ [Fig. 2(a)], and for both the values of α, we observe an unexpected behavior for the synchronization time when $\varepsilon>0.8$. We would expect that, having fixed all the other parameters, $\langle t_s\rangle$ were always a decreasing function of the interaction strength, because it would seem reasonable to think that greater values of the strength would tend to accelerate the synchronization. We have constructed several similar plots, not presented here, by changing the values of N and α. From such simulations, we could observe clearly that the "turning point" $\varepsilon=\varepsilon^{(1)}$, above which $\langle t_s\rangle$ starts to increase, depends only on N (of course, with α inside the synchronization domain) and it could be identified by the formula $\varepsilon^{(1)}=[\varepsilon_{up}(0,N)-\varepsilon_{lo}(0,N)]/2$. Of course, the values of N for which this atypical behavior is suppressed can be achieved by requiring that $\varepsilon^{(1)}\geqslant 1$. From the above formula, we can observe that this requirement will be satisfied for $N\geqslant N_{\min}=385$. This is numerically corroborated in Fig. 2(b), which shows the behavior of $\langle t_s\rangle$ for $N=N_{\min}$. These results indicate that the behavior of the synchronization time for $N=5$ is characteristic of small lattices. By the way, such a behavior was not observed when we varied the parameter α.

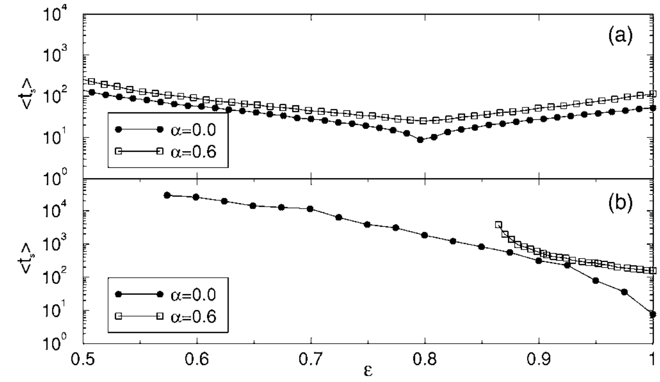

FIG. 2. Averaged synchronization time $\langle t_s\rangle$ vs ε, with $\alpha=0.0$ and $\alpha=0.6$. (a) $N=5$; (b) $N=385$.

In Figure 3(a), we plotted the averaged synchronization time with varying N, with both ε and α fixed. We can observe that $\langle t_s\rangle$ tends to saturate for large N. Moreover, the saturated time does not differ significantly from that corresponding to $N=N_{\min}$. This saturation is typical, i.e., still occurs even if we vary ε and α.

Now we consider the parameters of CML in an outer vicinity of the synchronization domain. The system no longer synchronizes unless it starts synchronized. Figure 4(a) is a typical plot for a time series of distances d_n in such a case. To make the visualization easier, we presented our results in terms of $y_n=-\log_{10}d_n$. Figure 4(b) shows the corresponding statistical distributions of y_n. Again the results suggest that, for large lattices, such a distribution is practically indepen-

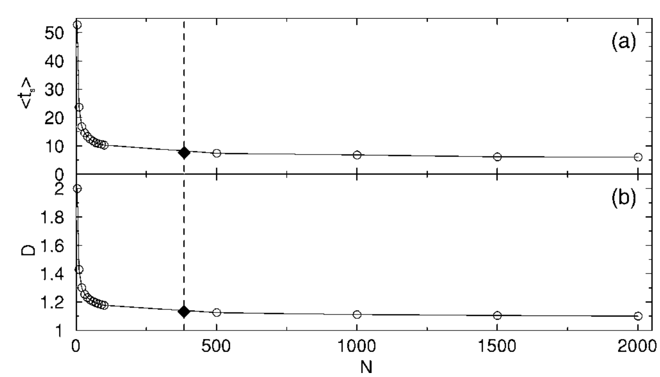

FIG. 3. (a) $\langle t_s\rangle$ vs N. (b) D vs N. In both plots $\alpha=0.0$ and $\varepsilon=1.0$. The vertical line corresponds to $N=385$.

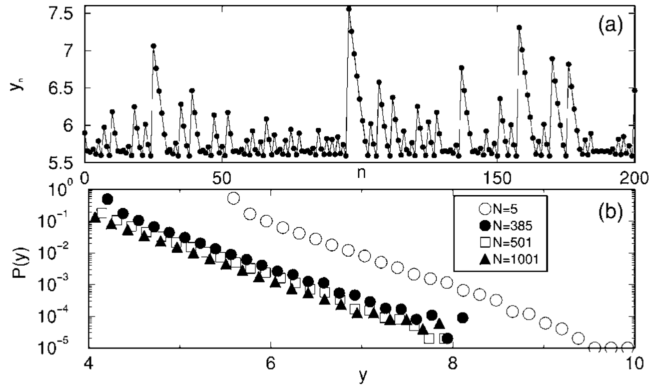

FIG. 4. (a) Time series for y_n with $N=385$, $\alpha=0.6+10^{-5}$, and $\varepsilon=\varepsilon_c(0.6,385)$. (b) Distributions of y for several values of N, with $\alpha=0.6+10^{-5}$ and $\varepsilon=\varepsilon_c(0.6,N)$.

dent of N and it is very well approximated by the distribution corresponding to $N=N_{\min}$. Additionally, we observe that these statistical distributions exhibit power law scalings with respect to the distance. Besides, taking into account the behavior of the second largest Lyapunov exponent in this outside region, we can infer that changes in the parameters α and ϵ affect only the slope of these distributions, which increases with increasing α and decreases with increasing ε.

All the results presented so far indicate that $N=N_{\min}=385$, apart from eliminating the aforementioned finite size qualitative effects of the synchronization time, also furnishes a reasonable quantitative approximation to the behavior of the system for larger N values, at least in what concerns the synchronization time and the statistical distribution of distances d_n. Such results motivate us to claim, at least in what concerns these two aspects, that $N_{\min}$ sets a practical lower bound in numerical simulations for the system to be considered at the thermodynamic limit. With this statement we mean that both the qualitative and quantitative behaviors of the system at the thermodynamic limit $N\to\infty$ can be reasonably well approximated by its corresponding behavior when $N=N_{\min}$.

Now, we recall that the time oscillation of d_n in Fig. 4(a) is due to the coexistence of both stable and unstable Lyapunov exponents in the direction transversal to the invariant subspace **S** [6]. Therefore, a closer examination of the Lyapunov spectrum could reveal some new aspects of the synchronization behavior. We thus consider the Lyapunov dimension of the system, which is a suitable concept to study the Lyapunov spectrum and is defined as follows. Let λ_j ($j=1,2,\ldots$) denote the jth largest Lyapunov exponent of the system and p be the largest integer for which $\sum_{j=1}^{p}\lambda_j$ is nonnegative. Then D is given by [3]

$$D=\begin{cases}0 & \text{if there is no such } p\\ p+\dfrac{1}{|\lambda_{p+1}|}\displaystyle\sum_{i=1}^{p}\lambda_i & \text{if } p<N\\ N & \text{if } p=N.\end{cases} \tag{3}$$

Given N, the Lyapunov spectrum and, by its turn, the Lyapunov dimension, can be analytically determined [7]. In Figure 5, we depict the Lyapunov dimension D versus the strength parameter ε, for $N=5$ and $N=385$, and for two values of α. We can observe two distinct behaviors for D as ε enters into the synchronization domain. For large N, the Lyapunov dimension monotonically decreases, but for small N there is a value $\varepsilon=\varepsilon^{(2)}$ above which D starts to increase. Figure 3(b) shows the dependence of D against N, for α and ϵ fixed within the synchronization domain. We can observe that the Lyapunov dimension tends to saturate with increasing N.

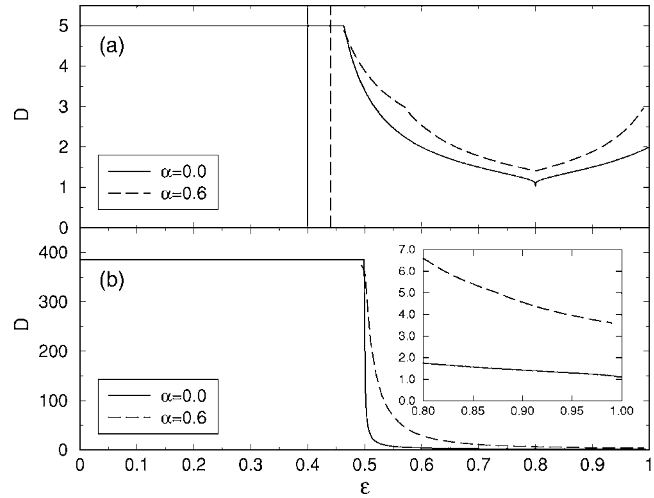

FIG. 5. D vs ε, with $\alpha=0.0$ and $\alpha=0.6$. (a) $N=5$; (b) $N=385$. The vertical lines indicate the boundary of the synchronization domain. Inset: detail of (b).

At this point we call attention to the great similarity between the behaviors of the averaged synchronization time $\langle t_s\rangle$ and the Lyapunov dimension within the synchronization domain. This similarity can be observed by directly comparing the shapes of Figs. 2 and 5 or by comparing Figs. 3(a) and 3(b). These plots suggest a correlation between the Lyapunov dimension and the averaged synchronization time.

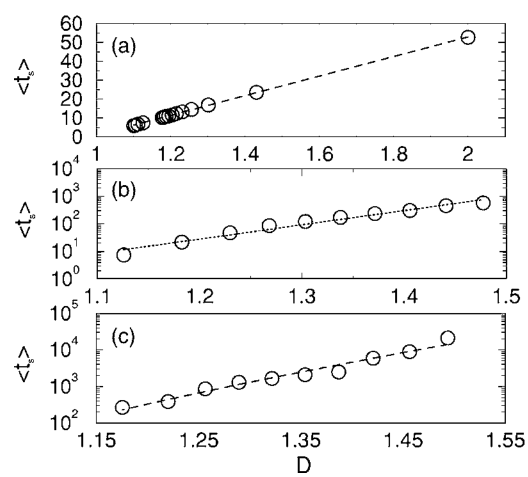

FIG. 6. Dispersion diagrams $\langle t_s\rangle$ vs D: (a) $\alpha=0.0$, $\varepsilon=1.0$, and $N\in[5,2000]$, $\rho=0.9998714$; (b) $\alpha=0.0$, $N=501$, and $\varepsilon\in[\varepsilon_c(0,501),1.0]$, $\rho=0.9849924$; (c) $N=501$, $\varepsilon=1.0$, and $\alpha\in[0.0,0.2]$, $\rho=0.9868354$.

In Fig. 6, we plot three dispersion diagrams $\langle t_s \rangle$ vs D, each one with two of the three parameters, α, ε, and N, fixed. All these diagrams give correlation coefficients ρ very close to 1, and they suggest a *very strong* correlation among these two quantities. In these plots the dashed lines correspond to the fitting functions, which are linear in the first case and are exponentials in the remaining two cases. The origin of such a strong correlation between so diversely defined quantities is a point that would need a deeper analysis. Nevertheless, we could try to understand this result on some intuitive grounds by observing that it is reasonable to think that the dominance of negative (positive) Lyapunov exponents in the direction transversal to the invariant subspace **S** would tend to minimize (maximize) the synchronization time. This is precisely the behavior of the Lyapunov dimension, as is obvious from its definition.

Summarizing, in this report we numerically simulated the behavior of a CML consisting of a chain of chaotic logistic maps exhibiting power law interactions. We observed size dependent behaviors with respect to the averaged synchronization time $\langle t_s \rangle$ and to the statistical distribution of distances d_n. Such behaviors motivated us to set the size $N_{\min}=385$ as a *practical* lower bound for this system to be considered in the thermodynamic limit in numerical simulations. We argued that the system behavior at the thermodynamic limit $N \rightarrow \infty$ can be reasonably well approximated, both qualitatively and quantitatively, by its behavior at this lower bound.

We also studied the behavior of the Lyapunov dimension of the system within the synchronization domain. Our results indicated the existence of a very strong correlation between this quantity and the averaged synchronization time. The origin of such a correlation and its related consequences are subjects that still need more clarifications and it will be postponed to future works. Additional studies concerning the scaling laws for the distribution of distances are now in progress. On the other hand, we are also considering other maps and interactions in (1). The results of these analyses will be presented elsewhere.

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